Int J Theor Phys (2007) 46: 2801–2807
DOI 10.1007/s10773-007-9391-5

Measure of Entanglement for General Pure Multipartite States Based on the Plücker Coordinates

Hoshang Heydari

Published online: 26 May 2007

Abstract We construct a measure of entanglement for general pure multipartite states based on the Plücker coordinates of the Grassmann variety. In particular, we step by step construct measures of entanglement for general pure bipartite, three-partite, four-partite, and m-partite states.

1 Introduction

In the field of quantum information theory it is very important to get a deep understanding of quantum entanglement. However, the quantification of multipartite entangled states is not an easy task and it usually relies on higher mathematical tools from different branches of mathematics. In recent years, there has been a lot of activity to quantify such entangled states and still this problem is not completely solved even for pure multipartite states. A linearly homogeneous positive functions of unnormalized pure states invariant under stochastic local quantum operation and classical communication (SLOCC) transformations give entanglement monotones [1]. The authors also have presented a general mathematical framework to describe local equivalence classes of multipartite quantum states under the action of local unitary and local filtering operations. Their analysis has lead to the introduction of entanglement measures for the multipartite states, and the optimal local filtering operations maximizing these entanglement monotones were obtained. Briand et al. [2] have studied the invariant theory of trilinear forms over a three-dimensional complex vector space, and applied it to investigate the behavior of pure entangled three-partite qutrit states. They described the orbit space of the SLOCC group $SL(3,\mathbf{C})^{\times 3}$ both in its affine and projective versions in terms of a very symmetric normal form parameterized by three complex numbers. They have also shown that the structure of the sets of equivalent normal forms is related to the geometry of certain regular complex polytopes. Miyake and Wadati [3] have

H. Heydari (✉)
Institute of Quantum Science, Nihon University, 1-8 Kanda-Surugadai, Chiyoda-ku,
Tokyo 101-8308, Japan
e-mail: hoshang@edu.cst.nihon-u.ac.jp

explored quantum search from the geometric viewpoint of a complex projective space. Recently, Lévay [4] have constructed a class of multi-qubit entanglement monotone which was based on construction of Emary [5]. His construction is based on bipartite partitions of the Hilbert space and the invariants are expressed in terms of the Plücker coordinates of the Grassmannian. We have also constructed entanglement monotone for multi-qubit states based on Plücker coordinate equations of the Grassmann variety, which are central notion in geometric invariant theory [6]. In this paper, we will construct a measure of entanglement for general multipartite states based on the complex algebraic variety. In particular, in Sect. 3 we will construct an entanglement measure for general bipartite states, which coincides with concurrence. In Sect. 4 we will construct a measure of entanglement for three-partite states which also coincides with generalized concurrence, and in Sect. 5, following the same recipe, we will construct a measure of entanglement for general four-partite states. Finally, in Sect. 6, we will generalize our result into the general pure multipartite states. Now, let us start by denoting a general, pure, composite quantum system with m subsystems as $\mathcal{Q} = \mathcal{Q}^p_m(N_1, N_2, \dots, N_m) = \mathcal{Q}_1\mathcal{Q}_2\cdots\mathcal{Q}_m$, consisting of the pure states $|\Psi\rangle = \sum_{k_1=1}^{N_1}\sum_{k_2=1}^{N_2}\cdots\sum_{k_m=1}^{N_m} \alpha_{k_1,k_2,\dots,i_m}|k_1, k_2, \dots, k_m\rangle$ and corresponding to the Hilbert space $\mathcal{H}_{\mathcal{Q}} = \mathcal{H}_{\mathcal{Q}_1} \otimes \mathcal{H}_{\mathcal{Q}_2} \otimes \cdots \otimes \mathcal{H}_{\mathcal{Q}_m}$, where the dimension of the jth Hilbert space is $N_j = \dim(\mathcal{H}_{\mathcal{Q}_j})$. We are going to use this notation throughout this paper. In particular, we denote a pure two-qubit state by $\mathcal{Q}^p_2(2, 2)$. For those readers who are unfamiliar with algebraic and projective variety we give a short introduction to these topics, however the standard references for the complex projective variety are [7, 8].

Let $\{f_1, f_2, \dots, f_q\}$ be continuous functions $\mathbf{C}^n \to \mathbf{C}$. Then we define a complex space as the set of simultaneous zeroes of the functions

$$\mathcal{V}_{\mathbf{C}}(f_1, f_2, \dots, f_q) = \{(z_1, z_2, \dots, z_n) \in \mathbf{C}^n : f_i(z_1, z_2, \dots, z_n) = 0,\ \forall 1 \le i \le q\}. \quad (1)$$

The complex space becomes a topological space by giving them the induced topology from $\mathbf{C}^n$. Now, if all f_i are polynomial functions in the coordinate functions, then the real (complex) space is called a real (complex) affine variety. A complex projective space $\mathbf{CP}^n$ is defined to be the set of lines through the origin in $\mathbf{C}^{n+1}$. That is, $\mathbf{CP}^n = (\mathbf{C}^{n+1} - 0)/\sim$, where $\sim$ is an equivalence relation define by $(x_1, \dots, x_{n+1}) \sim (y_1, \dots, y_{n+1}) \Leftrightarrow \exists\lambda \in \mathbf{C} - 0$, such that $\lambda x_i = y_i \forall\ 0 \le i \le n$. For $n = 1$ we have a one dimensional complex manifold $\mathbf{CP}^1$, which is a very important one, since as a real manifold it is homeomorphic to the 2-sphere $\mathbf{S}^2$, e.g., the Bloch sphere. Moreover, every complex compact manifold can be embedded in some $\mathbf{CP}^n$. In particular, we can embed a product of two projective spaces into the third one. Let $\{f_1, f_2, \dots, f_q\}$ be a set of homogeneous polynomials in the coordinates $\{\alpha_1, \alpha_2, \dots, \alpha_{n+1}\}$ of $\mathbf{C}^{n+1}$. Then the projective variety is defined to be the subset

$$\mathcal{V}(f_1, f_2, \dots, f_q) = \{[\alpha_1, \dots, \alpha_{n+1}] \in \mathbf{CP}^n : f_i(\alpha_1, \dots, \alpha_{n+1}) = 0,\ \forall 1 \le i \le q\}. \quad (2)$$

We can view the complex affine variety $\mathcal{V}_{\mathbf{C}}(f_1, f_2, \dots, f_q) \subset \mathbf{C}^{n+1}$ as a complex cone over the projective variety $\mathcal{V}(f_1, f_2, \dots, f_q)$.

2 Grassmann Variety

In this section, we will define the Grassmann variety in terms of the Plücker coordinate equations [9] which is necessary for understanding what follows in the following section, where we will construct measure of entanglement based on the Plücker coordinate of the

Grassmann variety. Let V be a complex vector space of dimension $d \geq 2$ and $0 < r < d$ be an integer. Then the Grassmannian $\mathrm{Gr}(r, d)$ is defined as the set of all r-dimensional subspaces of V, that is

$$\mathrm{Gr}(r, d) = \{W : W \text{ is a subspace of } V \text{ of dimension } r\}. \tag{3}$$

Alternatively, the Grassmannian $\mathrm{Gr}(r, d)$ can considered as the set of all $r-1$-dimensional linear projective subspaces of $\mathbf{CP}^{d-1}$. The simplest example of the Grassmannian is $\mathrm{Gr}(1, d)$ which is the set of all one dimensional subspaces of complex vector space V which is the complex projective space on V. Now, we can embed $\mathrm{Gr}(r, d)$ into $\mathbf{P}(\bigwedge^r(\mathbf{C}^d)) = \mathbf{CP}^{\mathcal{N}}$, $\mathcal{N} = \frac{d!}{(d-r)!r!} - 1$, by using the Plücker map $L \to \bigwedge^r(L)$, where L is the subspace to be embedded in the exterior product $\bigwedge^r(\mathbf{C}^d)$ for $1 \leq r \leq d$ which is a subspace of $\mathbf{C}^{N_1} \otimes \cdots \otimes \mathbf{C}^{N_m}$, spanned by the anti-symmetric tensors. The Plücker coordinates $P_{i_1,i_2,\ldots,i_r}$, $1 \leq i_1 < \cdots < i_r \leq d$ are the projective coordinates in this projective space. Next, let $\mathbf{C}[\Lambda(r, d)]$ be a polynomial ring with the Plücker coordinates P_J indexed by elements of the set $\Lambda(r, d)$ of ordered r-tuples in $\{1, 2, \ldots d\}$ as its variables. Then the image of the map

$$\kappa : \mathbf{C}[\Lambda(r, d)] \to \mathrm{Pol}(\mathrm{Mat}_{r,d}), \tag{4}$$

where $\mathrm{Mat}_{r,d}$ is a $r \times d$ matrix, which assigns to $P_{i_1 i_2 \ldots i_r}$ the bracket polynomial $[i_1, i_2, \ldots, i_r]$ is equal to the sub-ring of the invariant of the polynomials. The bracket function $[i_1, i_2, \ldots, i_r]$ on the given matrix $\mathrm{Mat}_{r,d}$ is equal to the maximal minor formed by the columns from a set of $\{1, 2, \ldots, d\}$. Alternatively, let $\{e_1, e_2, \ldots, e_d\}$ be a basis for complex vector space V. Then the canonical basis for $\bigwedge^r V$ is given by

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_r} \mid 1 \leq i_1 < \cdots \leq i_1 \leq d\}. \tag{5}$$

Now, let W be a r-dimensional subspace of V with basis $\{w_1, w_2, \ldots, w_r\}$ then the Plücker coordinates is given by $P_{i_1,i_2,\ldots,i_r}(W) = w_1 \wedge w_2 \wedge \cdots \wedge w_r$. Moreover, the kernel $\mathcal{I}_{r,d}$ of the map κ is equal to the homogeneous ideal of the Grassmann in its Plücker embedding. Furthermore, the homogeneous ideal $\mathcal{I}_{r,d}$ defining $\mathrm{Gr}(r, d)$ in its Plücker embedding is generated by the quadratic polynomials

$$\mathcal{P}_{I,J} = \sum_{t=1}^{r+1} (-)^t P_{i_1,\ldots,i_{r-1},j_t} P_{j_1\cdots j_{t-1} j_{t+1},\ldots,j_{r+1}}, \tag{6}$$

where $I = (i_1 \ldots i_{r-1})$, $1 \leq i_1 < \cdots < i_{r-1} < j_i$, for $i = 1, \ldots, r+1$, and $J = (j_1, \ldots, j_{r+1})$, $1 \leq j_1 < \cdots < j_{r+1} \leq d$ are two increasing sequences of numbers from the set $\{1, 2, \ldots, d\}$. Note that the equations $\mathcal{P}_{I,J} = 0$ defining the Grassmannian $\mathrm{Gr}(r, d)$ are called the Plücker coordinate equations. For example, for $\mathrm{Gr}(2, d)$ and $r = 2$, we have

$$\mathcal{P}_{I,J} = -P_{i_1,j_1} P_{j_2,j_3} + P_{i_1,j_2} P_{j_1,j_3} - P_{i_1,j_3} P_{j_1,j_2}, \tag{7}$$

where $I = (i_1)$, and $J = (j_1, j_2, j_3)$. Note that, by its construction, the Grassmannian $\mathrm{Gr}(2, d)$ is a homogeneous space with the Plücker coordinates are invariant under transformations of the form $S \otimes \cdots \otimes I \otimes I$, where $S \in SL(2, \mathbf{C})$ rotating the linearly independent basis vector in L. Moreover, there are many complex manifolds that cannot be embedded in complex projective space.

3 Measure of Entanglement for General Pure Bipartite States

The simplest composite quantum system is the bipartite states. So, in this section, we will concentrate on the construction of a measure of entanglement for general pure bipartite states based on Plücker coordinates of the Grassmann variety. Let us consider the quantum system $\mathcal{Q}_2^p(N_1, N_2)$. We now define a matrix

$$\mathrm{Mat}^1_{N_1,N_2} = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \dots & \alpha_{1,N_2} \\ \alpha_{2,1} & \alpha_{2,2} & \dots & \alpha_{2,N_2} \\ \vdots & \vdots & \dots & \vdots \\ \alpha_{N_1,1} & \alpha_{N_1,2} & \dots & \alpha_{N_1,N_2} \end{pmatrix}, \tag{8}$$

where, we have only 1 permutation by definition. This is very important to keep in mind when we construct an entanglement measure for multipartite states based on the Plücker coordinates of the Grassmann variety. In the case of multipartite states we need to apply this procedure for all possible permutation of indices. Now, let us consider a quantum system $\mathcal{Q}_2^p(2, N_2)$, where in this case we have $r = N_1 = 2$ and $d = N_2$. Then, we define

$$\begin{aligned} \mathcal{E}^2_{I,J}(\mathrm{Mat}^1_{N_1,N_2}(\nu,\mu)) &= \sum_{t=1}^{3}(P_1^{i_1,j_t}\overline{P}^1_{i_1,j_t} + P_1^{j_1\cdots j_{t-1}j_{t+1},\dots,j_3}\overline{P}^1_{j_1\cdots j_{t-1}j_{t+1},\dots,j_3}) \\ &= P_1^{i_1,j_1}\overline{P}^1_{i_1,j_1} + P_1^{j_2,j_3}\overline{P}^1_{j_2,j_3} + P_1^{i_1,j_2}\overline{P}^1_{i_1,j_2} \\ &\quad + P_1^{j_1,j_3}\overline{P}^1_{j_1,j_3} + P_1^{i_1,j_3}\overline{P}^1_{i_1,j_3} + P_1^{j_1,j_2}\overline{P}^1_{j_1,j_2}, \end{aligned} \tag{9}$$

where $\mathrm{Mat}^1_{N_1,N_2}$ is give by matrix (8). Thus, a measure of entanglement for general pure bipartite states based on equation (9) is given by

$$\begin{aligned} \mathcal{E}(\mathcal{Q}_2^p(N_1,N_2)) &= \left(\mathcal{N}_2 \sum_{\nu<\mu} \mathcal{E}^2_{I,J}(\mathrm{Mat}^1_{N_1,N_2}(\nu,\mu))\right)^{1/2} \\ &= \left(\mathcal{N}_2 \sum_{l_1>k_1=1}^{N_1} \sum_{l_2>k_2=1}^{N_2} |\alpha_{k_1,k_2}\alpha_{l_1,l_2} - \alpha_{k_1,l_2}\alpha_{l_1,k_2}|^2\right)^{1/2}, \end{aligned} \tag{10}$$

where $(\mathrm{Mat}^1_{N_1,N_2}(\nu,\mu))$ refers to rows ν and μ of matrix (8). This measure coincides with the generalized concurrence given in [10, 11]. However, this measure of entanglement which corresponds to rows $\nu < \mu$ of matrix (8) is invariant under action of $SL(2, \mathbf{C})$.

4 Measure of Entanglement for General Pure Three-Partite States

In this section, we will construct a measure of entanglement for general pure three-partite states based on the same method with which we have constructed a measure of entanglement for bipartite states. However, in the case of the bipartite states we only need to consider one matrix, that is, there is only one permutation, but for three-partite states we need to consider all possible permutations, namely three possible permutations of j_1, j_2, j_3. So, let us consider the quantum system $\mathcal{Q}_3^p(N_1, N_2, N_3)$. Then we define the matrices

$$\mathrm{Mat}^1_{N_1,N_2,N_3} = \begin{pmatrix} \alpha_{1,1,1} & \alpha_{1,1,2} & \cdots & \alpha_{1,N_2,N_3} \\ \alpha_{2,1,1} & \alpha_{2,1,2} & \cdots & \alpha_{2,N_2,N_3} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{N_1,1,1} & \alpha_{N_1,1,2} & \cdots & \alpha_{N_1,N_2,N_3} \end{pmatrix}, \tag{11}$$

$$\mathrm{Mat}^2_{N_1,N_2,N_3} = \begin{pmatrix} \alpha_{1,1,1} & \alpha_{1,1,2} & \cdots & \alpha_{N_1,1,N_3} \\ \alpha_{1,1,1} & \alpha_{1,2,2} & \cdots & \alpha_{N_1,2,N_3} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{1,N_2,1} & \alpha_{1,N_2,2} & \cdots & \alpha_{N_1,N_2,N_3} \end{pmatrix}. \tag{12}$$

The matrix $\mathrm{Mat}^3_{N_1,N_2,N_3}$ can also be constructed by permutation of indices of the matrix $\mathrm{Mat}^1_{N_1,N_2,N_3}$. Now, following the same recipe as in the case of the bipartite states, we can construct a measure of entanglement for general pure three-partite states as

$$\begin{aligned} &\mathcal{E}(\mathcal{Q}_3^p(N_1, N_2, N_3)) \\ &= \left(\mathcal{N}_3 \sum_{j=1}^{3} \sum_{\nu<\mu} \mathcal{E}^2_{I,J}(\mathrm{Mat}^j_{N_1,N_2,N_3}(\nu,\mu))\right)^{1/2} \\ &= \left(\mathcal{N}_3 \sum_{t=1}^{4} (P_j^{i_1,j_t} \overline{P}^j_{i_1,j_t} + P_j^{j_1\cdots j_{t-1} j_{t+1},\ldots,j_3} \overline{P}^j_{j_1\cdots j_{t-1} j_{t+1},\ldots,j_3})\right)^{1/2} \\ &= \left(\mathcal{N}'_3 \sum_{j=1}^{3} \left|\alpha_{k_1,k_2,k_3}\alpha_{l_1,l_2,l_3} - \alpha_{k_1,l_j,k_3}\alpha_{l_1,k_j,l_3}\right|^2\right)^{1/2} \end{aligned} \tag{13}$$

where e.g., $(\mathrm{Mat}^1_{N_1,N_2,N_3}(\nu,\mu))$ refers to rows $\nu < \mu$ of matrix (11). The measure of entanglement for three-partite states (13) coincides with generalized concurrence [11] and has a well-defined geometrical interpretation in terms of the Plücker coordinates of the Grassmannian. This geometrical illustration can gives us some intuition about the properties of multipartite entanglement in general. Moreover, this measure is an entanglement monotone for three-qubit states.

5 Measure of Entanglement for General Pure Four-Partite States

Next, we will construct a measure of entanglement for general pure four-partite states following the same recipe as in the case of three-partite states. However, the difference in this case compares to three-partite states is that we have more than two sets of permutations. So, let us first look at the first set of permutations which is given by permutations of the following matrix

$$\mathrm{Mat}^1_{N_1,N_2,N_3,N_4} = \begin{pmatrix} \alpha_{1,1,1,1} & \alpha_{1,1,1,2} & \cdots & \alpha_{1,N_2,N_3,N_4} \\ \alpha_{2,1,1,1} & \alpha_{2,1,1,2} & \cdots & \alpha_{2,N_2,N_3,N_4} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{N_1,1,1,1} & \alpha_{N_1,1,1,2} & \cdots & \alpha_{N_1,N_2,N_3,N_4} \end{pmatrix}. \tag{14}$$

That is, $\mathrm{Mat}^{j}_{N_1,N_2,N_3,N_4}$, $j = 2, 3, 4$ can be constructed in the same way as in the case of three-partite states by permutation of indices of matrix (14). The second set of permutations can be constructed by simultaneous permutations of two of the indices of the matrix

$$\mathrm{Mat}^{1,2}_{N_1,N_2,N_3,N_4} = \begin{pmatrix} \alpha_{1,1,1,1} & \alpha_{1,1,1,2} & \cdots & \alpha_{1,1,N_3,N_4} \\ \alpha_{2,2,1,1} & \alpha_{2,2,1,2} & \cdots & \alpha_{2,2,N_3,N_4} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{N_1,N_2,1,1} & \alpha_{N_1,N_2,1,2} & \cdots & \alpha_{N_1,N_2,N_3,N_4} \end{pmatrix}. \tag{15}$$

Again, the matrix $\mathrm{Mat}^{k,l}_{N_1,N_2,N_3,N_4}$ for $1 = k < l = 2, 3, 4$ can be constructed by permutations of matrix (15). For example, for quantum system $\mathcal{Q}^p_4(2, 2, 2, 2)$, we have

$$\begin{aligned} \mathrm{Mat}^{1,2}_{2,2,2,2} &= \begin{pmatrix} \alpha_{1,1,1,1} & \alpha_{1,1,1,2} & \alpha_{1,1,2,1} & \alpha_{1,1,2,2} \\ \alpha_{2,2,1,1} & \alpha_{2,2,1,2} & \alpha_{2,2,2,1} & \alpha_{2,2,2,2} \end{pmatrix}, \\ \mathrm{Mat}^{1,3}_{2,2,2,2} &= \begin{pmatrix} \alpha_{1,1,1,1} & \alpha_{1,1,1,2} & \alpha_{1,2,1,1} & \alpha_{1,2,1,2} \\ \alpha_{2,1,2,1} & \alpha_{2,1,2,2} & \alpha_{2,2,2,1} & \alpha_{2,2,2,2} \end{pmatrix}. \end{aligned} \tag{16}$$

Thus, for four-partite states, we need to consider to different types of matrices which we have constructed by different sets of permutations. Now, we can construct a measure of entanglement for a general pure four-partite state as

$$\begin{aligned} \mathcal{E}(\mathcal{Q}^p_4(N_1, N_2, N_3, N_4)) = \Bigg(\mathcal{N}_4 \Bigg[\sum_{j=1}^{4} \sum_{\nu<\mu} \mathcal{E}^2_{I,J}(\mathrm{Mat}^{j}_{N_1,N_2,N_3,N_4}(\nu, \mu)) \\ + \sum_{k<l} \sum_{\nu<\mu} \mathcal{E}^2_{I,J}(\mathrm{Mat}^{k,l}_{N_1,N_2,N_3,N_4}(\nu, \mu)) \Bigg] \Bigg)^{1/2}, \end{aligned} \tag{17}$$

where, e.g., $(\mathrm{Mat}^{1}_{N_1,N_2,N_3,N_4}(\nu, \mu))$ refers to rows $\nu < \mu$ of matrix (14) and $(\mathrm{Mat}^{1,2}_{N_1,N_2,N_3,N_4}(\nu, \mu))$ refers to rows $\nu < \mu$ of matrix (15). For multi-qubit states, this measure of entanglement is an entanglement monotone by construction.

6 Measure of Entanglement for General Pure Multipartite States

In this section, we would like to generalize our result from four-partite states to m-partite states in a straightforward manner. For example, the first set of permutations is given by permutations of the matrix

$$\mathrm{Mat}^{1}_{N_1,N_2\cdots N_m} = \begin{pmatrix} \alpha_{1,1,\ldots,1} & \alpha_{1,1,\ldots,2} & \cdots & \alpha_{1,N_2,\ldots,N_m} \\ \alpha_{2,1,\ldots,1} & \alpha_{2,1,\ldots,2} & \cdots & \alpha_{2,N_2,\ldots,N_m} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha_{N_1,1,\ldots,1} & \alpha_{N_1,1,\ldots,2} & \cdots & \alpha_{N_1,N_2,\ldots,N_m} \end{pmatrix}. \tag{18}$$

All other sets of permutations can be constructed by permutations of the matrix $\mathrm{Mat}^{p_1,p_2,\ldots,p_m}_{N_1,N_2,\ldots,N_m}$ for all $(p_1, p_2, \ldots, p_m) \in \sigma(j_1, j_2, \ldots, j_m)$, where $\sigma(j_1, j_2, \ldots, j_m)$ denotes the sets of all possible permutations of $j_1, j_2, \ldots, j_m$ as we have illustrated in the cases of three- and four-partite states. Next, we can construct a measure of entanglement for general pure multi-partite states as

$$\mathcal{E}(\mathcal{Q}_4^p(N_1,N_2,\dots,N_4)) = \left(\mathcal{N}_4 \sum_{\forall(p_1,\dots,p_m)\in\sigma(j_1,\dots,j_m)} \sum_{\nu<\mu} \mathcal{E}_{I,J}^2(\mathrm{Mat}_{N_1,\dots,N_m}^{p_1,\dots,p_m}(\nu,\mu))\right)^{1/2}, \quad (19)$$

where, e.g., $(\mathrm{Mat}^1_{N_1,N_2,\dots,N_m}(\nu,\mu))$ refers to rows $\nu<\mu$ of matrix (18) from the first set of permutations of $j_1, j_2, \dots, j_m$. Again, by construction this measure of entanglement is an entanglement monotone for multi-qubit states.

7 Conclusion

In this paper, we have constructed a measure of entanglement for multipartite states based on the Grassmannian $\mathrm{Gr}(r,d)$, which was defined in terms of the Plücker coordinate equations. By construction our measure vanishes on product states and it is invariant under actions of the special linear group. In particular, we have explicitly constructed measures of entanglement for bipartite states, three-partite states and four-partite states in a concrete way that can be directly applied to any such general pure state. The measure of entanglement coincides with the concurrence for bipartite and three-partite states, which can be seen directly from the expression and our measure of entanglement is an entanglement monotone for multi-qubit states. Finally, we hope that our result can give some geometrical insight to solving such interesting problem of the fundamental quantum theory with wide application in emerging field of the quantum information and quantum computing.

Acknowledgements The author gratefully acknowledges the financial support of the Japan Society for the Promotion of Science (JSPS).

References

1. Verstraete, F., Dehaene, J., De Moor, B.: Phys. Rev. A **68**, 012103 (2003)
2. Briand, E., Luque, J.-G., Thibon, J.-Y., Verstraete, F.: J. Math. Phys. **45**, 4855 (2004), quant-ph/0306122
3. Miyake, A., Wadati, M.: Quantum Inf. Comput. **2**, 540–555 (2002)
4. Lévay, P.: J. Phys. A: Math. Gen. **38**, 9075–9085 (2005)
5. Emary, C.: J. Phys. A: Math. Gen. **37**, 8293 (2004)
6. Heydari, H.: J. Math. Phys. **47**, 012103 (2006)
7. Griffiths, P., Harris, J.: Principle of Algebraic Geometry. Wiley, New York (1978)
8. Mumford, D.: Algebraic Geometry I: Complex Projective Varieties. Springer, Berlin (1976)
9. Mumford, D., Fogarty, J., Kirwan, F.: Geometric Invariant Theory, 3rd edn. Springer, Berlin (1994)
10. Gerjuoy, E.: Phys. Rev. A **67**, 052308 (2003)
11. Albeverio, S., Fei, S.M.: J. Opt. B: Quantum Semiclass. Opt. **3**, 223 (2001)